ON GAS FLOW IN LAVAL NOZZLES

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Let us consider the flow of an ideal gas in the neighborhood of the surface of transition from subsonic to supersonic flow in a Laval nozzle, which has two planes of symmetry. The straight line of intersection of these two planes we will call the axis of the nozzle, while the point of intersection of the nozzle axis with the sonic transition surface, to which the axis is normal, is the center of the nozzle. Making the origin of cylindrical coordinates x, r, θ coincident with the center of the nozzle and choosing the x-axis to be coincident with the axis of the nozzle, let us write the equation, which determined the gas flow in the neighborhood of the surface of transition, in the form

$$-\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}=0 \qquad (1)$$

where ϕ is the potential, such that

$$\frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial x}=v_x, \qquad \frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial r}=v_r, \qquad \frac{a_*}{\varkappa+1}\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}=v_\vartheta \qquad (2)$$

where v_x, v_x, v_θ are perturbations, in the x, r, θ directions, on the velocity which in magnitude is equal to the critical velocity a_* and is directed along the nozzle axis; κ is Poisson's adiabatic index.

In order to get shock-free solutions of the nozzle, let us consider only analytical solutions of the equation (1). In the case of plane and round nozzles the desired solutions have the form: [2,3]

$$\varphi=r^4 f(\xi,\vartheta), \qquad \xi=x/r^2 \qquad (3)$$

In the general case we will look for solutions of the equation of motion (1) in the foregoing similarity form. Substituting formulas (3) into equation (1) we get:

$$\left(4\xi^2-\frac{\partial f}{\partial\xi}\right)\frac{\partial^2 f}{\partial\xi^2}-12\,\xi\frac{\partial f}{\partial\xi}+\frac{\partial^2 f}{\partial\vartheta^2}+16f=0 \qquad (4)$$

Since we are interested only in analytical solutions of equation (1), the desired solutions of equation (4) are of the form:

$$f = \frac{A}{2}\xi^2 + g_1(\vartheta)\xi + g_2(\vartheta) \tag{5}$$

Using equation (4), we will get the following expressions for functions $g_1(\theta)$ and $g_2(\theta)$:

$$g_1(\vartheta) = A^2\left(\frac{1}{4} - n\cos 2\vartheta\right)$$

$$g_2(\vartheta) = A^3\left(\frac{1}{64} - \frac{n}{12}\cos 2\vartheta + m\cos 4\vartheta\right) \tag{6}$$

where A, n and m are arbitrary constants; in what follows we will take $A > 0$ everywhere. From formulas (2), (3), (5) and (6) we find the expressions for the potential and the velocity components of the flow:

$$\varphi = \frac{A}{2}x^2 + A^2\left(\frac{1}{4} - n\cos 2\vartheta\right)xr^2 + A^3\left(\frac{1}{64} - \frac{n}{12}\cos 2\vartheta + m\cos 4\vartheta\right)r^4$$

$$\frac{\varkappa+1}{a_*}v_x = Ax + A^2\left(\frac{1}{4} - n\cos\vartheta\right)r^2$$

$$\frac{\varkappa+1}{a_*}v_r = A^2\left(\frac{1}{2} - 2n\cos 2\vartheta\right)xr + A^3\left(\frac{1}{16} - \frac{n}{3}\cos 2\vartheta + 4m\cos 4\vartheta\right)r^3 \tag{7}$$

$$\frac{\varkappa+1}{a_*}v_\vartheta = 2nA^2xr\sin 2\vartheta + A^3\left(\frac{n}{6}\sin 2\vartheta - 4m\sin 4\vartheta\right)r^3$$

In the above solution the functions v_x and v_r are even with regard to θ, the function v_θ is odd. Hence it follows that formulas (7) describe the flow in the neighborhood of the surface of transition from subsonic velocities to supersonic velocities in nozzles whose cross-section has two axes of symmetry.

Assuming that $n = m = 0$ in formulas (7), we obtain a stream in a round Laval nozzle; choosing $n = \pm 1/4$, $m = 1/192$, we have the flow in a plane nozzle [1-3].

Let us consider now the form of the surface of transition through the velocity of sound, which is obtained from the equality $v_x = 0$. Hence we have, using the second of formulas (7),

$$-\xi = A\left(\frac{1}{4} - n\cos 2\vartheta\right)$$

Going over from the cylindrical coordinate system to Cartesian coordinates by substituting $z = r\cos\theta$, $y = r\sin\theta$, we obtain the equation of the transition surface in the form

$$-x = A\left(\frac{1}{4} + n\right)y^2 + A\left(\frac{1}{4} - n\right)z^2 \tag{8}$$

From equation (8) it follows that for $|n| < 1/4$ the surface of transition will be an elliptical paraboloid, for $|n| = 1/4$ it is a parabolic cylinder, for $|n| > 1/4$ a hyperbolic paraboloid. From relations (7) it is easy to see that it is possible to give the magnitude of the velocity of the stream, and, consequently, the form of the transition surface corresponding to flow in plane and round nozzles even though the flow as a whole is not given. We also note that in the general case the surfaces $v_r = 0$ $v\theta = 0$ may not coincide; the former is described by the equation

$$\xi = -A\,\frac{{}^1/_{16} - {}^1/_3\, n\cos 2\vartheta + 4m\cos 4\vartheta}{{}^1/_2 - 2n\cos 2\vartheta}$$

the latter is given by

$$\xi = A\,\frac{-{}^1/_6\, n\sin 2\vartheta + 4m\sin 4\vartheta}{2n\sin 2\vartheta}$$

Let us find the equations of the characteristic surfaces passing through the center of the nozzle. The characteristic surfaces are described by the first order differential equation

$$\left(\frac{\partial x}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial x}{\partial \vartheta}\right)^2 - Ax = A^2\left(\frac{1}{4} - n\cos 2\vartheta\right)r^2 \qquad (9)$$

which can be formulated in the form $\text{grad}^2\, x = \partial\phi/\partial x$. Hence it follows immediately that as the characteristic surfaces approach the surface of transition through the velocity of sound they become perpendicular to the axis of the nozzle. Therefore the characteristic surfaces touch the surface of transition at the point $x = 0$, $r = 0$. We will consider only those characteristic surfaces which have the same planes of symmetry as the nozzle itself, which pass through its center and do not have breaks. For their determination we will assume:

$$x = \chi(\vartheta)\, r^2 \qquad (10)$$

Substituting expression (10) into equation (9), we obtain a first order ordinary differential equation for the function $\chi(\theta)$:

$$\left(\frac{d\chi}{d\vartheta}\right)^2 = -4\chi^2 + A\chi + \frac{A^2}{4} - A^2 n\cos 2\vartheta \qquad (11)$$

For the characteristic surfaces under consideration the derivative $d\chi/d\theta$ must be equal to zero for $\theta = 0$ and $\theta = \frac{1}{2}\pi$, because in this case a function $x(r, \theta)$, determined in the first quadrant of the plane $r\theta$, can be continued symmetrically, as follows from equation (11), into the remaining three quadrants without discontinuities of the first derivative with respect to θ. This can be obtained only for $|n| < 5/16$. The desired solutions of equation (11) can be formulated as follows:

$$\chi = \frac{A}{16}\,[2 \pm \Delta_1 \pm \Delta_2 + (\pm\Delta_1 \mp \Delta_2)\cos 2\vartheta]$$

$$\chi = \frac{A}{16}\,[2 \pm \Delta_1 \mp \Delta_2 + (\pm\Delta_1 \pm \Delta_2)\cos 2\vartheta]$$

where $\Delta_1 = \sqrt{5 - 16\,n}$, $\Delta_2 = \sqrt{5 + 16\,n}$. Four characteristic equations, touching the sonic transition surface at the center of the nozzle, can be now written in Cartesian coordinates in the form

$$x = \frac{1}{8} A (1 \pm \Delta_2) y^2 + \frac{1}{8} A (1 \pm \Delta_1) z^2$$
$$x = \frac{1}{8} A (1 \mp \Delta_2) y^2 + \frac{1}{8} A (1 \pm \Delta_1) z^2 \qquad (12)$$

For $|n| < 1/4$ the first two of equations (12) constitute the equations of elliptical paraboloids, extending in opposite directions along the nozzle axis, while the second two are hyperbolic paraboloids. For $n = 1/4$ equations (12) give

$$x = \frac{1}{2} Ay^2 + \frac{1}{4} Az^2, \qquad x = -\frac{1}{4} Ay^2 + \frac{1}{4} Az^2, \qquad x = -\frac{1}{4} Ay^2, \qquad x = \frac{1}{2} Ay^2$$

For $n = -1/4$ we obtain correspondingly:

$$x = \frac{1}{4} Ay^2 + \frac{1}{2} Az^2, \qquad x = \frac{1}{2} Az^2, \qquad x = -\frac{1}{4} Az^2, \qquad x = \frac{1}{4} Ay^2 - \frac{1}{4} Az^2$$

Thus, for $|n| = 1/4$ two of the surfaces under consideration constitute parabolic cylinders, one is an elliptical paraboloid and one a hyperbolic paraboloid. For $|n| > 1/4$ two of the characteristic surfaces, given by formulas (12), are elliptical paraboloids, extending along the nozzle axis in the direction of increasing values of x, the other two are hyperbolic paraboloids. For $|n| = 5/16$ elliptical and hyperbolic paraboloids coincide in pairs. We may note that the first of the solutions (12) describes a disturbance caused by a needle situated on the axis of the nozzle and touching with its point the surface of transition; in this case the Mach cone transforms into an elliptical paraboloid.

For $1/4 < n < 5/16$ the two elliptical paraboloids given by equations (12) are tangent to each other along a curve lying in the plane $z = 0$. Between these paraboloids it is possible to construct a family of characteristic surfaces which have breaks on the z-axis, in cross-sections x = constant, and which touch along the indicated tangent curve of the characteristic paraboloids. For $-5/16 < n < -1/4$ analogous characterist surfaces will have breaks on the y-axis in cross-sections x = const. Characteristic surfaces for $|1/4| < n < |5/16|$, situated downstream, which are tangent in the center of the nozzle to the surface of transition and which do not have any other common points with the outside characteristic paraboloid, except the point $x = 0$, $r = 0$, vanish. For $|n| > 5/16$, there does not exist a characteristic surface which does not touch the sonic transition surface anywhere, except at the center of the nozzle, and which extends downstream (i.e. corresponding to the limiting Mach "cone" with apex at the nozzle center). The latter statements are easily obtained if equation (9) is written in the form:

$$\frac{1}{r}\frac{\partial x}{\partial \vartheta} = \pm \sqrt{Ax + A^2\left(\frac{1}{4} - n\cos 2\vartheta\right) r^2 - \left(\frac{\partial x}{\partial r}\right)^2}$$

and the expression under the radical in this formula is analysed.

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